SECTION 15.1: FUNCTIONS OF SEVERAL VARIABLES

RECALL: A **function** is a relation that takes a given input and returns **one and only one** output. We have seen **real-valued** functions (functions which take **real numbers** to **real numbers**) and **vector-valued functions** (functions which take **real numbers** to **vectors**.) Here we begin the study of functions of several variables which we can interpret as taking **points** (in 2, 3, or more dimensions) to **real numbers**.

EXAMPLE 1: Let $f(x,y) = 2xy^2$. Find and simplify:

1. $f(-1,2)$

   Ans: $f(-1,2) = -8$

2. $f(2,-1) =$

   Ans: $f(2,-1) = 4$

3. $f(t,2t) =$

   Ans: $f(t,2t) = 8t^3$

4. $f(y,x) =$

   Ans: $f(y,x) = 2x^2y$

5. $f(x+h,y) =$

   Ans: $f(x+h,y) = 2y^2h + 2xy^2$

6. $f(x+h,y) - f(x,y)$

   Ans: $f(x+h,y) - f(x,y) = 2y^2h$

7. $\dfrac{f(x+h,y) - f(x,y)}{h} =$

   Ans: $\dfrac{f(x+h,y) - f(x,y)}{h} = 2y^2$

EXAMPLE 2: What goes wrong when trying to find the indicated function value for the functions listed below?

1. Find $f(0,0)$ for $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

Ans: 0 in denominator.

2. Find $f(-2,4)$ for $f(x,y) = \dfrac{3x^2}{2x+y}$.

Ans: 0 in denominator.

3. Find $f(4,5)$ for $f(x,y) = \sqrt{25-x^2-y^2}$.

Ans: $(-)$ under square root.

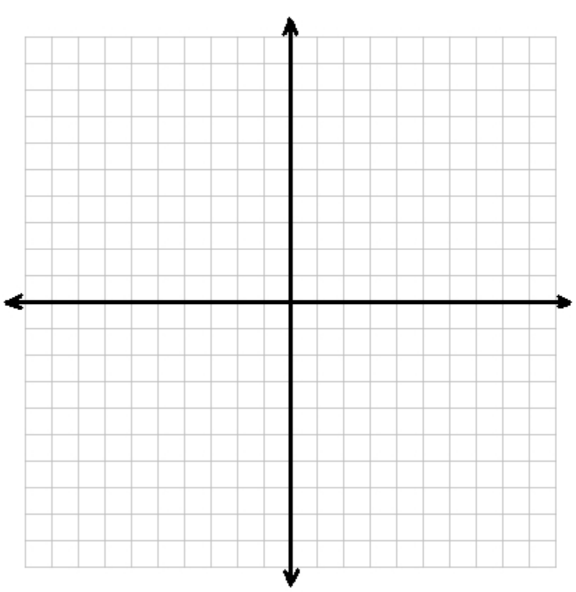
4. Find $f(3,9)$ for $f(x,y) = \ln\left(y-x^2\right)$.

Ans: 0 in logarithm.

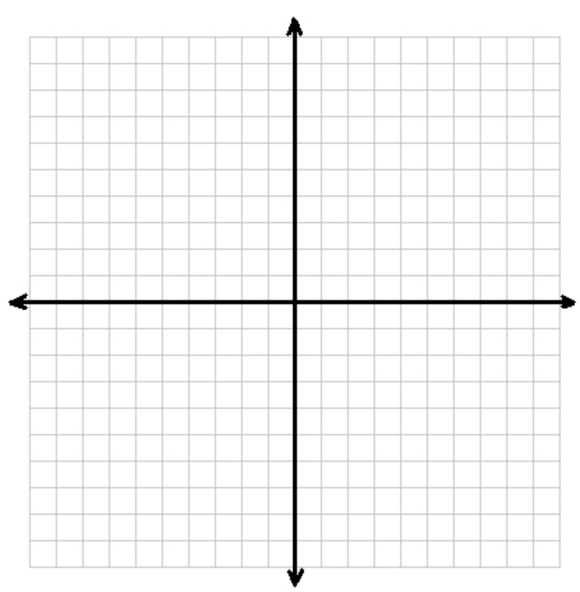
RECALL: The **domain** of a function is the set if allowable **inputs**.

EXAMPLE 3: Sketch the domain of the following functions in the plane.

1. $f(x,y) = \dfrac{2xy}{x^2+y^2}$.



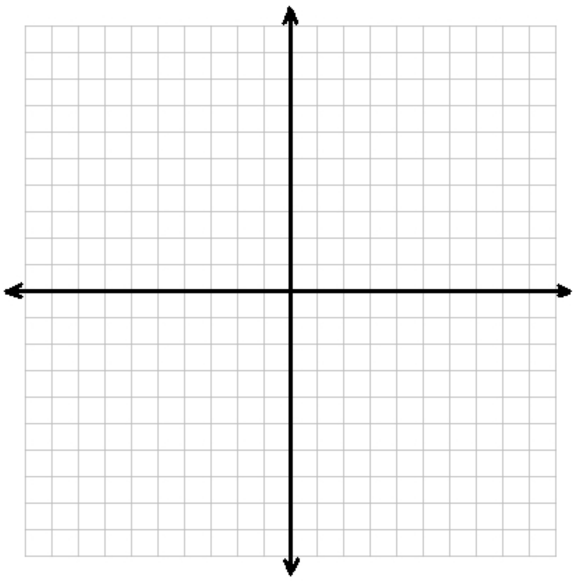
2. $f(x,y) = \dfrac{3x^2}{2x+y}$.

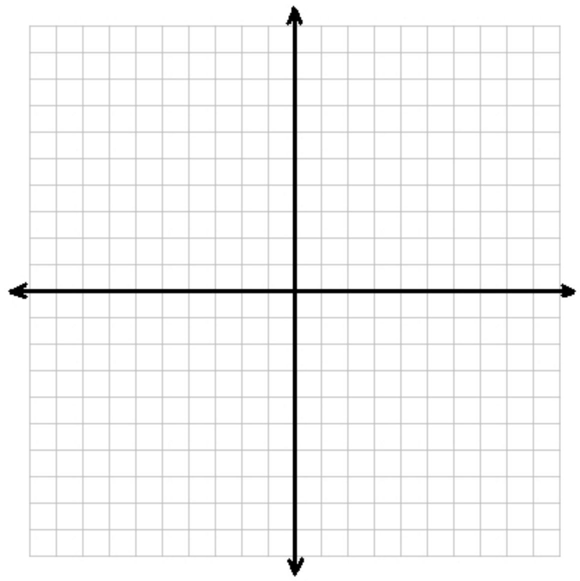


EXAMPLE 4: Sketch the domain of the following functions in the plane.

1. $f(x, y) = \sqrt{25 - x^2 - y^2}$.

2. $f(x, y) = \ln\left(y - x^2\right)$.





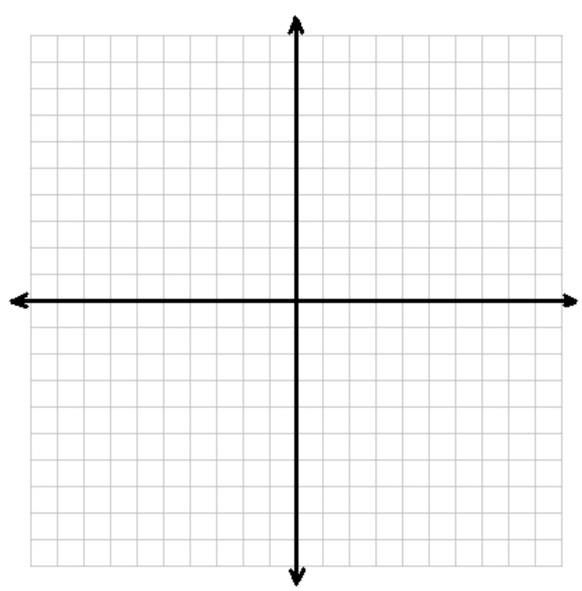
GRAPHS OF FUNCTIONS OF TWO VARIABLES

DEFINITION: To graph a function of two variables $f(x, y)$, we graph the **equation** $z = f(x, y)$. In this case, we say the variables x and y are **independent variables** while the variable z is called the **dependent** variable.

NOTE: 'Usually,' the graph of $z = f(x, y)$ is surface, so we may use all of the techniques we learned in the last chapter to help us visualize the graph including **traces**, **contours** along with more general **slices**.

Since here $z = f(x, y)$, making a 'contour map' by slicing with horizontal planes of the form $z = c$ amounts to graphing so-called ' **level curves**' corresponding to $f(x, y) = c$ for constants c in the range of f.

EXAMPLE 5: Make a contour map for $f(x, y) = \sqrt{25 - x^2 - y^2}$ by graphing $f(x, y) = c$ for $c = 0$, 3, 4, 5.



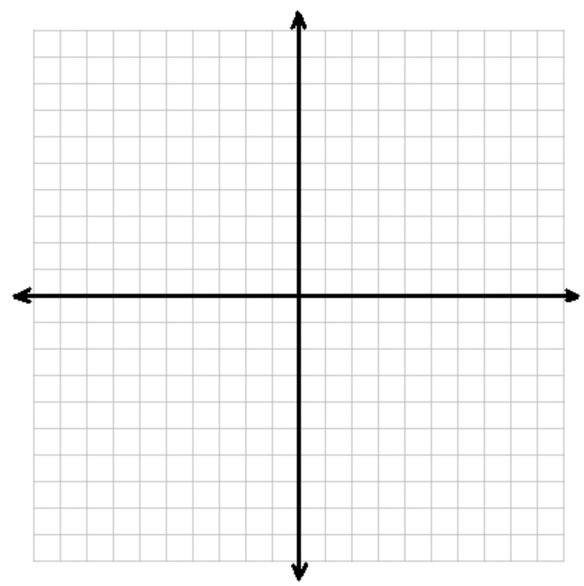
EXAMPLE 6: The profit in **thousands** of dollars from selling x **hundred** phones and y **hundred** tablets is:

$$P(x, y) = -x^2 - y^2 + 20x + 10y - 25, \quad x, y \geq 0$$

1. Find $P(4, 5)$ and interpret what this means in terms of phone and tablet sales and profit.

Ans: $P(4, 5) = 64$. This means when 400 phones and 500 tablets are sold, the profit is \$64,000.

2. Graph the contour corresponding to $P(x, y) = P(4, 5)$.



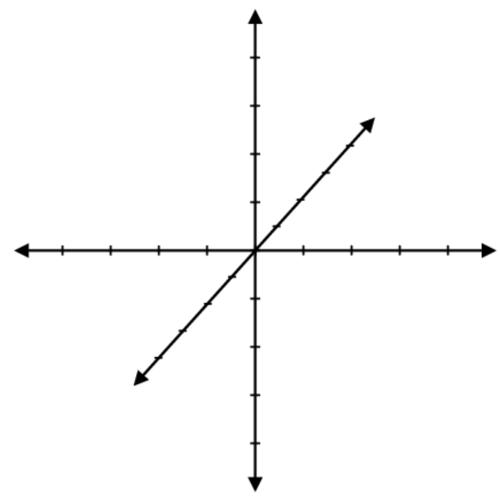
If a point (a, b) is on this contour, what does it mean in terms of phone and tablet sales and profit?

(a, b) is on the contour means selling a hundred phones and b hundred tablets gives a profit of \$64,000.

VISUALIZING FUNCTIONS OF THREE VARIABLES

EXAMPLE 7: Let $F(x,y,z)=\sqrt{25-x^2-y^2-z^2}$.

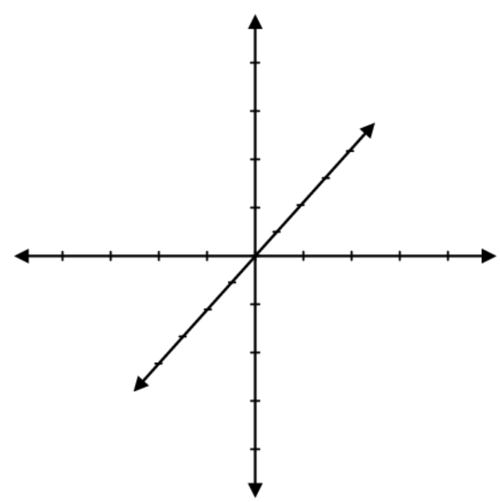
1. Sketch or otherwise describe the domain of F in space:



2. Find $F(1,-2,2)$.

Ans: $F(1,-2,2)=4$

3. Sketch the level **surface** $F(x,y,z)=F(1,-2,2)$.



What does it mean for a point (a,b,c) to be on the level surface $F(x,y,z)=F(1,-2,2)$?

Ans: If (a,b,c) is on the surface $F(x,y,z)=F(1,-2,2)$, it means $F(a,b,c)=F(1,-2,2)=4$.

EXAMPLE 8: Use GeoGebra to investigate level surfaces $F(x, y, z) = c$ for $-10 \leq c \leq 10$:

Write a sentence or two about what you observe.

1. $F(x, y, z) = x + y + z$

2. $F(x, y, z) = x^2 + y^2 + z^2$

3. $F(x, y, z) = x^2 + y^2 - z^2$

4. $F(x, y, z) = x^2 - y^2 - z^2$

HOMEWORK: Section 15.1: 9 - 81 every other odd